



12.5 Powers of matrices

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Introduction

- In this topic, we will
 - Discuss calculating $A^k \mathbf{u}_0$ for different values of k
 - See how using an eigenvalue decomposition aids us in this calculation
 - Observe that this results in fewer computations, and therefore, both faster and with less numeric error
 - See that we can use this approach even if A is not self-adjoint
 - Not symmetric if it is a real matrix, and not conjugate symmetric if it is a complex matrix





Powers of a matrix

- In some situations, a matrix may represent a transition from one state to the next
 - If A is such a matrix, and $\mathbf{u}_0$ is the initial state, then $\mathbf{u}_1 \leftarrow A\mathbf{u}_0$ is the next state
 - In general, if $\mathbf{u}_k$ is the current state, $\mathbf{u}_{k+1} \leftarrow A\mathbf{u}_k$ is the next
 - Alternatively, we could make a direct calculation: $\mathbf{u}_k \leftarrow A^k\mathbf{u}_0$
 - Issues:
 - Matrix exponentiation is expensive
 - There are many floating-point operations, so numeric error can quickly affect the results





Powers of a matrix

- Now, if A can be diagonalized,
then $A = UDU^{-1}$

– In this case, note that

$$\begin{aligned} A^2 &= (UDU^{-1})^2 \\ &= (UDU^{-1})(UDU^{-1}) \\ &= UD(U^{-1}U)DU^{-1} \end{aligned}$$

– Note, however, $U^{-1}U = I$

$$\begin{aligned} &= UDDU^{-1} \\ &= UD^2U^{-1} \end{aligned}$$





Powers of a matrix

- In general, therefore, if $A = UDU^{-1}$,
it follows that:

$$A^n = U D^n U^{-1}$$

- What are the benefits?
 - Calculating a diagonal matrix to the n^{th} power is trivial:
 - It simply requires us to raise each diagonal entry to the power n
 - If all the eigenvalues are non-zero, the matrix is invertible, and

$$A^{-1} = U D^{-1} U^{-1}$$





Powers of a matrix

- For example, consider the following symmetric matrix:

$$A = \begin{pmatrix} 0.5 & 0.1 & 0.1 & 0.1 & 0.2 \\ 0.1 & 0.2 & 0.1 & 0.3 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0 & 0.4 \\ 0.1 & 0.3 & 0 & 0.6 & 0 \\ 0.2 & 0.3 & 0.4 & 0 & 0.1 \end{pmatrix}$$

- Calculate $A^{10}\mathbf{u}_0$ for

$$\mathbf{u}_0 = \begin{pmatrix} 0.2 \\ 0.7 \\ 0.1 \\ 0.5 \\ -0.8 \end{pmatrix}$$





Powers of a matrix

- We find that

$$A = \begin{pmatrix} 0.447 & 0.129 & -0.874 & -0.111 & 0.081 \\ 0.447 & -0.168 & 0.198 & -0.476 & -0.712 \\ 0.447 & 0.498 & 0.396 & -0.360 & 0.515 \\ 0.447 & -0.778 & 0.128 & 0.171 & 0.386 \\ 0.447 & 0.319 & 0.152 & 0.776 & -0.270 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.648 & 0 & 0 & 0 \\ 0 & 0 & 0.383 & 0 & 0 \\ 0 & 0 & 0 & -0.298 & 0 \\ 0 & 0 & 0 & 0 & 0.068 \end{pmatrix} U^T$$

- Thus,

$$D^{10} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1.31 \times 10^{-2} & 0 & 0 & 0 \\ 0 & 0 & 6.73 \times 10^{-5} & 0 & 0 \\ 0 & 0 & 0 & 5.58 \times 10^{-6} & 0 \\ 0 & 0 & 0 & 0 & 1.99 \times 10^{-12} \end{pmatrix}$$





Powers of a matrix

- Therefore,

$$A^{10} \begin{pmatrix} 0.2 \\ 0.7 \\ 0.1 \\ 0.5 \\ -0.8 \end{pmatrix} = U \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1.31 \times 10^{-2} & 0 & 0 & 0 \\ 0 & 0 & 6.73 \times 10^{-5} & 0 & 0 \\ 0 & 0 & 0 & 5.58 \times 10^{-6} & 0 \\ 0 & 0 & 0 & 0 & 1.99 \times 10^{-12} \end{pmatrix} U^T \begin{pmatrix} 0.2 \\ 0.7 \\ 0.1 \\ 0.5 \\ -0.8 \end{pmatrix}$$

$$= \begin{pmatrix} 0.13885134679 \\ 0.14150442106 \\ 0.13553208893 \\ 0.14697571819 \\ 0.13713642503 \end{pmatrix}$$





Which approach to use?

- Suppose we want to calculate all values

$$\mathbf{u}_0, \mathbf{u}_1 = A\mathbf{u}_0, \mathbf{u}_2 = A^2\mathbf{u}_0, \mathbf{u}_3 = A^3\mathbf{u}_0, \dots, \mathbf{u}_{10} = A^{10}\mathbf{u}_0$$

- Question:

- Do we calculate $\mathbf{u}_1 \leftarrow A\mathbf{u}_0$, then $\mathbf{u}_2 \leftarrow A\mathbf{u}_1$, ...
and finally, then $\mathbf{u}_{10} \leftarrow A\mathbf{u}_9$?

- Or, do we calculate $\mathbf{u}_1 \leftarrow UDU^{-1}\mathbf{u}_0$
 $\mathbf{u}_2 \leftarrow UD^2U^{-1}\mathbf{u}_0$
 $\vdots$

$$\mathbf{u}_{10} \leftarrow UD^{10}U^{-1}\mathbf{u}_0$$

- Issue: there are many calculations involved with $\mathbf{u}_{k+1} \leftarrow A\mathbf{u}_k$
 - Errors introduced with previous calculations are magnified
- Using the eigenvalue decomposition,
calculating D^n is easy and relatively error free for all n





Avoiding the inverse

- We've mentioned before:
 - don't calculate the inverse unless it is straight-forward
 - The inverse of a diagonal matrix is that diagonal matrix with the reciprocals of the diagonal entries
 - The inverse of an isometric operator is its adjoint
 - For a real orthogonal matrix, the inverse is the transpose
 - For a complex unitary matrix,
the inverse is the conjugate transpose
 - Question:
 - What if the matrix is neither of these?





Avoiding the inverse

- We can perform these calculations without explicit calculations of the matrix inverse
 - Suppose that A is not self-adjoint; that is,
 - If A is a real matrix, it is not symmetric,
 - If A is a complex matrix, it is not conjugate symmetric
 - In calculate $\mathbf{v} = A^n \mathbf{u}$, assuming we can diagonalize the matrix, we calculate $\mathbf{v} = (UD^nU^{-1})\mathbf{u} = U(D^n (U^{-1}\mathbf{u}))$
 - Let $\mathbf{y} = U^{-1}\mathbf{u}$, and therefore $U\mathbf{y} = \mathbf{u}$
 - Solve the system of linear equations for $\mathbf{y}$
 - Finally, calculate $\mathbf{v} = U(D^n\mathbf{y})$





Summary

- Following this topic, you now
 - Understand how to use the eigenvalue decomposition to calculate $A^k \mathbf{u}_0$ for different values of k
 - Understand that this results in more accurate answers, with often fewer computations
 - This is especially the case if we only want to calculate the occasional $A^k \mathbf{u}_0$
 - Understand that, as with *PLU* decomposition, we can still calculate this without explicitly calculating the inverse of the matrix





References

- [1] https://en.wikipedia.org/wiki/Diagonalizable_matrix#Diagonalization





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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